Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; RICHARD EDEN, student, Purdue University, West Lafayette, IN, USA; OLEH FAYNSHTEYN, Leipzig, Germany; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and PETER Y. WOO, Biola University, La Mirada, CA, USA. As usual, Stan Wagon verified the validity of the inequality by using "FindInstance" (in 13 seconds).

3549. [2010 : 241, 243] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that $\left(1+a^2b\right)\left(1+b^2c\right)\left(1+c^2a\right) \leq 5+3abc$.

Solution by Arkady Alt, San Jose, CA, USA.

Note first that the given inequality is equivalent to

$$a^2b+b^2c+c^2a+abc(ab^2+bc^2+ca^2)+a^3b^3c^3 \leq 4+3abc\,. \qquad (1)$$

We first establish the following lemma:

Lemma *If a, b, and c are nonnegative real numbers, then*

$$a^2b+b^2c+c^2a+abc \leq \frac{4}{27}(a+b+c)^3\,. \qquad (2)$$

Proof: Due to the cyclic symmetry of a, b, and c in (2) we may assume, without loss of generality, that $c=\min\{a,b,c\}$. We consider two cases separately:

Case (i). Suppose $b \leq a$. By the AM-GM Inequality, we have

$$\frac{4}{27}(a+b+c)^3 = \frac{1}{2}\left(\frac{2b+2(a+c)}{3}\right)^3 \geq \frac{1}{2}\left(2b(a+c)^2\right) = b(a+c)^2\,. \qquad (3)$$

Since

$$\begin{aligned} b(a+c)^2-(a^2b+b^2c+c^2a+abc) &= abc+bc^2-b^2c-c^2a \\ &= c(ab+bc-b^2-ca) \\ &= c(a-b)(b-c) \geq 0 \end{aligned}$$

we have

$$a^2b+b^2c+c^2a+abc \leq b(a+c)^2 \qquad (4)$$

and (2) follows from (3) and (4).

Case (ii). Suppose $b > a$. We have

$$\begin{aligned} 2(a^2b+b^2c+c^2a+abc) &= \sum_{cyclic}(a^2b+ab^2)+2abc+\sum_{cyclic}(a^2b-ab^2) \\ &= (a+b)(b+c)(c+a)-(a-b)(b-c)(c-a)\,. \qquad (5) \end{aligned}$$

By the AM-GM Inequality we have

$$\begin{aligned}(a+b)(b+c)(c+a) &\leq \left(\frac{(a+b)+(b+c)+(c+a)}{3}\right)^3 \\ &= \frac{8}{27}(a+b+c)^3\end{aligned}$$

so

$$\begin{aligned}\frac{4}{27}(a+b+c)^3 &\geq \frac{1}{2}(a+b)(b+c)(c+a) \\ &= a^2b+b^2c+c^2a+abc+\frac{1}{2}(a-b)(b-c)(c-a) \\ &\geq a^2b+b^2c+c^2a+abc\end{aligned}$$

since $(a-b)(b-c)(c-a) \geq 0$.

This completes the proof of the lemma.

Since $a+b+c=3$, (2) becomes $a^2b+b^2c+c^2a \leq 4-abc$ and since $4-abc$ is invariant under the interchanging of a and b, we have $\max\{a^2b+b^2c+c^2a, ab^2+bc^2+ca^2\} \leq 4-abc$. Therefore,

$$\sum_{cyclic} a^2b + abc\sum_{cyclic} ab^2 + a^3b^3c^3 \leq (1+abc)(4-abc)+a^3b^3c^3\,.$$

Finally, since $abc \leq \left(\dfrac{a+b+c}{3}\right)^3 = 1$ we have

$$\begin{aligned}4+3abc-\left(\sum_{cyclic} a^2b + abc\sum_{cyclic} ab^2 + a^3b^3c^3\right) & \\ \geq 4+3abc-(1+abc)(4-abc)-a^3b^3c^3 & \\ = a^2b^2c^2-a^3b^3c^3 = a^2b^2c^2(1-abc) \geq 0 &\end{aligned}$$

which establishes (1) and completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer. Stan Wagon gave his usual verification using "FindInstances".